

# Super-inflation and generation of first order vector perturbations in ELKO

**Abhishek Basak and S. Shankaranarayanan**

School of Physics, Indian Institute of Science Education and Research Thiruvananthapuram (IISER-TVM), Trivandrum 695016, India

E-mail: [abhishek@iisertvm.ac.in](mailto:abhishek@iisertvm.ac.in), [shanki@iisertvm.ac.in](mailto:shanki@iisertvm.ac.in)

**Abstract.** In this work we construct a model where the first order vector perturbations can be generated during inflationary expansion. For the non-standard spinors, known as ELKO, we show that the components of the first order perturbed energy-momentum tensor of the ELKO is non-zero for pure vector part of the metric perturbation ( $B_i$ ). We show that vector perturbations do not decay in the super-Hubble scale and for a specific super-inflation background model we show that the vector perturbations are nearly scale invariant, while its amplitude is smaller than the primordial scalar perturbations. We also comment on the generation of vorticity.

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## 1 Introduction

Inflationary paradigm has been highly successful in explaining the observed Universe. In the standard inflationary scenario, inflation is driven by a slowly rolling scalar field through its potential. The cosmological perturbation theory during inflation has predicted the CMB observations quite successfully, for example density perturbation or scalar perturbations in the first order [1, 2].

However, generation of primordial seed magnetic field is still unresolved. There are various mechanisms to generate primordial magnetic field [3–5]. One mechanism for the generation of primordial magnetic field is to generate vorticity during inflation which can be sourced by primordial vector modes. However, it has been observed that the first order perturbation theory can not generate growing vector perturbations. The reason is easily understood as the non-diagonal component of perturbed energy-momentum tensor do not contain any vector modes. In the absence of anisotropic stress, the vector modes decay quickly as the Universe expands [1, 6]. To avoid this problem of first order vector modes attempts have been made to generate vector modes in the collapsing Universe during the contracting phase of the cyclic models of the Universe [7]. It has been shown that during this contracting phase the vector modes indeed grow, but this growth cannot be stopped which may finally lead to breakdown of the perturbation theory [8].

Unlike the first order perturbation, it has been observed that in the case of second order perturbation theory the vector modes can be sourced and vorticity can be generated [9–12] even in the standard scalar field driven inflationary theory.

In this work we consider the inflationary scenario driven by non-standard spinors also known as *ELKO* [13–16]. These kind of spinors have mass dimension one and follows Klein-Gordon equation instead of Dirac equation. It has been shown that this kind of spinors can drive inflation and lead to scalar power spectrum consistent with the observed data [17–19]. Here we show explicitly that ELKO driven inflation can generate growing vector modes even in the first order. We show that, unlike the scalar field inflation, the non-diagonal components of the stress-tensor (specifically  $(\eta, i)$  component, where  $\eta$  is the conformal time

defined later and  $i$  the spatial index) is non-zero corresponding to the pure vector modes of the metric perturbation. Rewriting the non-diagonal components, we show that the vector perturbations, like the scalar perturbations, satisfy second order differential equation. In order to make a definite prediction, we assume that the background field leads to super-inflation, i.e.  $\dot{H} > 0$  [20–23], where  $H$  is the Hubble parameter. The super-inflationary phase requires new physics which, in our case, is provided by ELKO. We have shown that the vector modes can be frozen in the super-Hubble scale and is scale invariant. However, the amplitude is small compared to the scalar perturbation. The suppression factor of the amplitudes of the vector modes is  $\exp\left(-\frac{9}{8}\Delta N\right)$ , where  $\Delta N$  is the number of e-foldings necessary for super-inflation. As in the case of ELKO driven inflationary theories, the spectral index of the scalar modes of perturbations depends on the slow-roll parameter  $\epsilon = -\frac{\dot{H}}{H^2}$ , super-inflationary phase ( $\dot{H} > 0$ ) may produce blue tilt in the spectral index of the scalar perturbations [19]. Hence, the super-inflationary phase precedes the standard inflationary phase. In order for the scalar perturbations from the ELKOS to be consistent with the CMB observations, the number of e-foldings in the super-inflation phase can only be of the order unity. After the end of super-inflationary phase the standard inflationary phase with ( $\dot{H} < 0$ ) starts.

In Sec. (2), we give the definitions of ELKO Lagrangian and the energy-momentum tensor. In this section we further calculate the non-diagonal component (specifically,  $(\eta, i)$ ) of the perturbed energy-momentum tensor in the linear order. In Sec. (3) we use the linear order perturbed Einstein equation and using the perturbed energy-momentum tensor we calculate the evolution equation of the pure vector modes  $B_i$ . At super-Hubble scale, it is shown that the vector modes can be frozen in time. In Sec. (4) we give the approximate background scaling solution, consistent with the super-inflation. Sec. (5) contains the solutions of the vector modes in the sub-Hubble and super-Hubble scale. Finally, in Sec. (6) we end with conclusions and comment on the generation of vorticity in the first order.

## 2 Linear order perturbed ELKO energy-momentum tensor

In this work we are interested in the inflationary theory based on non-standard spinor known as ELKO. ELKO are the eigenspinors of charge conjugation operator. This kind of spinors are non-standard because, unlike classical Dirac spinors in case of ELKO spinors  $(CPT)^{-2} = -\mathbb{I}$ . Other major difference between classical Dirac spinors and ELKO is, Dirac spinors have mass dimension  $\frac{3}{2}$  whereas ELKO spinors have mass dimension one. Therefore, these kind of spinors follow second order Klein-Gordon equation instead of Dirac equation which is first order in time. The energy-momentum tensor of ELKO can be written as [18]:

$$T^{\mu\nu} = \bar{\lambda} \overleftarrow{\nabla}^{(\mu} \overrightarrow{\nabla}^{\nu)} \lambda - g^{\mu\nu} \mathcal{L} + F^{\mu\nu}, \quad (2.1)$$

where  $\lambda$  and  $\bar{\lambda}$  are ELKO and its dual, respectively and  $\mu$  is the space-time index. The dual of ELKO is defined in the same spirit as is done in case of Dirac spinors (Dirac adjoint in case of Dirac spinors,  $\bar{\psi} = \psi^\dagger \gamma^0$ ,  $\gamma^0$  is the 0-th component of Dirac gamma matrix), such that  $(\bar{\lambda}\lambda)$  becomes a space-time scalar. For a detailed discussion on the construction of ELKO and its dual, one can look into [18]. The ELKO Lagrangian ( $\mathcal{L}$ ) is

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \bar{\lambda} \overleftarrow{\nabla}^{(\mu} \overrightarrow{\nabla}^{\nu)} \lambda - V(\bar{\lambda}\lambda) \quad (2.2)$$

The exact form of the potential is arbitrary at this stage as it will be clear in the following sections that it does not enter in the equation of motion of the vector modes directly. The covariant derivative on ELKO and its dual are defined as:

$$\overleftarrow{\nabla} \bar{\lambda} \equiv \partial^\mu \bar{\lambda} + \bar{\lambda} \Gamma^\mu, \quad \overrightarrow{\nabla} \lambda \equiv \partial^\mu \lambda - \Gamma^\mu \lambda. \quad (2.3)$$

Where  $\Gamma_\mu$  is the spin-connection appearing because of propagation of spinors in curved space-time. The expression of  $\Gamma_\mu$  is given by

$$\Gamma_\mu = \frac{i}{4} \omega_\mu^{ab} f_{ab}, \quad (2.4)$$

where  $a, b$  are the spinor indices.  $f^{ab} = \frac{i}{2} [\gamma^a, \gamma^b]$  is the generator of the Lorentz group and  $\gamma^a$  is the Dirac gamma matrix.  $\omega_\mu^{ab}$  is defined as  $\omega_\mu^{ab} = e_\alpha^a (e^{\alpha b})_{;\mu}$ . Here  $(; \mu)$  denotes the covariant derivative with respect to  $\mu$ .  $e_a^\mu$  is the vierbiens. The vierbiens are related to the space-time metric  $g^{\mu\nu}$  by the following relation:

$$e_a^\mu e_b^\nu \eta^{ab} = g^{\mu\nu}, \quad (2.5)$$

where  $\eta^{ab} = \text{diag.}(1, -1, -1, -1)$  is the Minkowski metric.

$F^{\mu\nu}$  is the additional term that comes from the variation of spin-connection,  $\Gamma^\mu$ , with respect to the metric  $g^{\mu\nu}$ . We work in the conformal time where the background metric is given as:

$$g_{\mu\nu}^{(0)} = \begin{pmatrix} a^2(\eta) & \mathbb{O} \\ \mathbb{O} & -a^2(\eta) \delta_{ij} \end{pmatrix}. \quad (2.6)$$

Here  $\eta$  is the conformal time defined as  $d\eta = dt/a$ ,  $t$  is the cosmic time and  $a$  is the scale factor. The expression of  $F^{\mu\nu}$  is given as

$$F^{\mu\nu} = \frac{1}{2} \nabla_\rho J^{\mu\nu\rho}, \quad (2.7)$$

where the expression of  $J^{\mu\nu\rho}$  is:

$$J^{\mu\nu\rho} = -\frac{i}{2} \left[ \bar{\lambda} \overleftarrow{\nabla}^{(\mu} f^{\nu)\rho} \lambda + \bar{\lambda} f^{\rho(\mu} \overrightarrow{\nabla}^{\nu)} \lambda \right]. \quad (2.8)$$

$f^{\mu\nu}$  is given as  $f^{\mu\nu} = e_a^\mu e_b^\nu f^{ab}$ .

We prefer to work with the following ansatz for the back ground ELKO and its dual

$$\lambda = \varphi(\eta) \xi, \quad \bar{\lambda} = \varphi(\eta) \bar{\xi}, \quad (2.9)$$

where  $\varphi$  is real,  $\xi$  and  $\bar{\xi}$  are the two constant matrices with the property

$$\bar{\xi} \xi = \mathbb{I}, \quad (2.10)$$

such that  $\bar{\lambda} \lambda = \varphi(\eta)^2$ . In the above ansatz  $\varphi(\eta)$  is a background scalar quantity dependent on time only. The advantages of using the above ansatz are: (i) The components of the energy-momentum tensor can be written in terms of one scalar field ( $\varphi$ ) instead of two spinors and (ii) it can be ensured that the theory does not have any negative energy or ghost modes [18]. Using (2.9) and (2.10) one can show that the  $(\eta, \eta)$  component of the background energy-momentum tensor, which will be used later, becomes:

$$T^{\eta\eta(0)} = a^{-4} \left[ \frac{1}{2} \varphi'^2 + \frac{3}{8} \mathcal{H}^2 \varphi^2 + a^2 V \right], \quad (2.11)$$

where  $'$  denotes derivative with respect to  $\eta$  and  $\mathcal{H}$  is the Hubble parameter defined as  $\mathcal{H} = a'/a$ .

The perturbed Einstein equation is given by:

$$\delta G_\nu^\mu = 8\pi G \delta T_\nu^\mu, \quad (2.12)$$

For the vector modes in the metric perturbation we choose the Newtonian gauge where the components of metric perturbation are given below:

$$\delta g_{\eta i} = a^2 B_i, \quad (2.13)$$

where ' $i$ ' is the spatial index  $-x, y$  and  $z$ . All other components of the metric perturbation are equal to zero. Here  $B_i$  is the divergence less vector mode i.e.,  $\partial^i B_i = 0$ . The  $(\eta, i)$  component of the Einstein equation (2.12) is given by,

$$\frac{1}{2}a^{-2}\Delta B_i = 8\pi G \delta T_i^\eta, \quad (2.14)$$

where  $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ .

It is important to note that in the standard scalar field driven inflation the right hand side of the equation (2.14) vanishes. Therefore, in case of the standard scalar field theory the solution of the vector modes  $B_i$  are identically zero in all scales. But here we will show that unlike the standard canonical scalar field theory, in case of ELKO the expression of  $\delta T_i^\eta$  has the vector modes  $B_i$ .

To calculate the perturbed energy momentum tensor we use the following ansatz for the perturbed ELKO and its dual:

$$\delta \lambda = \delta \varphi(\eta, \vec{x}) \xi, \quad \delta \bar{\lambda} = \delta \varphi(\eta, \vec{x}) \bar{\xi}. \quad (2.15)$$

The expression of energy-momentum tensor (2.1) shows that it is a sum of three components:  $T1^{\mu\nu} = \bar{\lambda} \overleftrightarrow{\nabla}^{(\mu} \nabla^{\nu)} \lambda$ ,  $T2^{\mu\nu} = g^{\mu\nu} \mathcal{L}$  and  $T3^{\mu\nu} = F^{\mu\nu}$ . Here we show the  $(\eta, x)$  component of the three terms of the perturbed energy-momentum tensor. The other components  $((\eta, y)$  and  $(\eta, z))$  can be written accordingly. The expression of  $\delta T1^{\eta x}$ ,  $\delta T2^{\eta x}$  and  $\delta T3^{\eta x}$  can be written as follows:

$$\delta T1^{\eta x} = a^{-4} \left[ B_x \varphi'^2 - \varphi' \delta \varphi_{,x} - \frac{1}{8} \mathcal{H} B_x' \varphi^2 \right], \quad (2.16)$$

$$\delta T2^{\eta x} = a^{-4} \left[ \frac{1}{2} B_x \varphi'^2 + \frac{3}{8} \mathcal{H}^2 B_x \varphi^2 - a^2 B_x V \right], \quad (2.17)$$

$$\delta T3^{\eta x} = a^{-4} \left[ \frac{3}{4} \mathcal{H}^2 B_x \varphi^2 - \frac{1}{16} B_x'' \varphi^2 - \frac{1}{8} B_x' \varphi \varphi' + \frac{1}{4} \mathcal{H} \varphi \delta \varphi_{,x} + \frac{1}{8} \Delta B_x \varphi^2 \right]. \quad (2.18)$$

In the above expression  $(_{,x})$  denotes the partial derivative with respect to  $x$ . Therefore, one can generalize  $(\eta, i)$  component of the energy-momentum tensor for vector perturbation as:

$$\begin{aligned} \delta T^{\eta i} = a^{-4} & \left[ \frac{1}{2} B_i \varphi'^2 - \varphi' \delta \varphi_{,i} - \frac{1}{8} \mathcal{H} B_i' \varphi^2 + \frac{3}{8} \mathcal{H}^2 B_i \varphi^2 + a^2 B_i V - \frac{1}{16} B_i'' \varphi^2 - \frac{1}{8} B_i' \varphi \varphi' + \right. \\ & \left. \frac{1}{4} \mathcal{H} \varphi \delta \varphi_{,i} + \frac{1}{8} \Delta B_i \varphi^2 \right] \end{aligned} \quad (2.19)$$

The expression of energy-momentum tensor in the mixed form can be calculated using the following relation:

$$\delta T_\nu^\mu = \delta T^{\mu\sigma} g_{\sigma\nu}^{(0)} + T^{\mu\sigma(0)} \delta g_{\sigma\nu}. \quad (2.20)$$

Therefore, using the expression (2.11)  $(\eta, i)$  component of the energy-momentum tensor in the mixed form becomes:

$$\delta T_i^\eta = a^{-2} \left[ \frac{1}{8} \mathcal{H} B_i' \varphi^2 + \frac{1}{16} B_i'' \varphi^2 + \frac{1}{8} B_i' \varphi \varphi' + \left( \varphi' - \frac{1}{4} \mathcal{H} \varphi \right) \delta \varphi_{,i} - \frac{1}{8} \Delta B_i \varphi^2 \right] \quad (2.21)$$

### 3 Evolution equation of $B_i$

Substituting equation (2.21) in the equation (2.14) the expression of the Einstein equation can be written as:

$$\Delta B_i \left[ 1 + \frac{1}{4} \frac{\varphi^2}{M_{pl}^2} \right] = \frac{1}{M_{pl}^2} \left[ \frac{1}{4} \mathcal{H} B_i' \varphi^2 + \frac{1}{8} B_i'' \varphi^2 + \frac{1}{4} B_i' \varphi \varphi' + 2 \left( \varphi' - \frac{1}{4} \mathcal{H} \varphi \right) \delta \varphi_{,i} \right], \quad (3.1)$$

where  $M_{pl} = \frac{1}{\sqrt{8\pi G}}$  is the reduced Planck Mass. The above equation can be rewritten as

$$B_i'' + 2 \left( \frac{\varphi'}{\varphi} + \mathcal{H} \right) B_i' - \left( 2 + \frac{8M_{pl}^2}{\varphi^2} \right) \Delta B_i + 16 \left( \varphi' - \frac{1}{4} \mathcal{H} \varphi \right) \frac{\delta \varphi_{,i}}{\varphi^2} = 0, \quad (3.2)$$

In the Fourier mode the equation (3.1) can be written as

$$B_i'' + A_1 B_i' + A_2 k^2 B_i - 16 i k_i \left( \varphi' - \frac{1}{4} \mathcal{H} \varphi \right) \frac{\delta \varphi}{\varphi^2} = 0, \quad (3.3)$$

where,  $A_1 = 2 \left( \frac{\varphi'}{\varphi} + \mathcal{H} \right)$  and  $A_2 = 2 + \frac{8M_{pl}^2}{\varphi^2}$ . The Fourier modes are related to the partial derivatives as  $\partial_i \equiv -i k_i$ . The last term of the above equation acts as a source term. For simplicity, one can set the last term in Eq. (3.3) to vanish, i.e.,

$$\left( \varphi' - \frac{1}{4} \mathcal{H} \varphi \right) = 0. \quad (3.4)$$

Under this condition, the evolution equation (3.2) simplifies to

$$B_i'' + A_1 B_i' - A_2 \Delta B_i = 0. \quad (3.5)$$

The above equation looks very similar to the evolution equation of scalar perturbations during inflation. In the next section we show that the condition (3.4) leads to consistent background evolution.

### 4 Background scaling solution

Before proceeding with the power-spectrum calculation, we show that the condition (3.4) leads to a consistent background evolution and that it leads to Super-inflation ( $\dot{H} > 0$ ).

The Klein-Gordon equation of the background field  $\varphi$  is given as [18]

$$\varphi'' + 2\mathcal{H}\varphi' - \frac{3}{4}\mathcal{H}^2\varphi + a^2V_{,\varphi} = 0, \quad (4.1)$$

where  $(,_{\varphi})$  denotes the derivative with respect to the background field  $\varphi$ . The modified Friedmann equations are given as follows:

$$\mathcal{H}^2 = \frac{1}{1 - \tilde{F}} \left[ \frac{1}{3M_{\text{pl}}^2} \left( \frac{\varphi'^2}{2} + a^2V \right) \right], \quad (4.2)$$

$$\mathcal{H}' = \frac{1}{1 - \tilde{F}} \left[ \frac{1}{3M_{\text{pl}}^2} (a^2V - \varphi'^2) + \mathcal{H}\tilde{F}' \right], \quad (4.3)$$

where  $\tilde{F} = \frac{\varphi^2}{8M_{\text{pl}}^2}$ . To find the background scaling solutions for power-law type of potential we choose the following forms of scale factor, background field and potential:

$$a(\eta) = A(-\eta)^{-q}, \quad \varphi(\eta) = \varphi_0(-\eta)^p, \quad V = V_0\varphi^\beta, \quad (4.4)$$

where  $A$ ,  $\varphi_0$  and  $V_0$  are some arbitrary constants which can be expressed in terms of the exponents  $-q$ ,  $p$  and  $\beta$  using the three background equations. Keeping in mind the condition that  $\varphi' = \frac{1}{4}\mathcal{H}\varphi$  one can easily write  $p = -nq$ , where  $n = \frac{1}{4}$ . The equations (4.2) and (4.3) are qualitatively same. Substituting (4.4) in the background equations (4.1) and (4.2) one find the following relations between  $\beta$  and  $q$  respectively

$$q = 2/(2 + n\beta - 2n), \quad q = 2/(2 + n\beta). \quad (4.5)$$

Therefore, one can consistently solve the background equations for

$$\beta \gg 2, \quad 0 < q \ll 1. \quad (4.6)$$

Writing the background equation (4.3) in cosmic time gives us:

$$\dot{H} (1 - \pi G \varphi^2) = -4\pi G \dot{\varphi}^2 + 2\pi G H \varphi \dot{\varphi}, \quad (4.7)$$

where  $H = \mathcal{H}/a$ . From the above expression one can see that when  $\frac{\dot{\varphi}}{H\varphi} > \frac{1}{2}$  one gets the standard inflationary theory with  $\dot{H} < 0$  and when  $\frac{\dot{\varphi}}{H\varphi} < \frac{1}{2}$  one gets the super-inflationary theory with  $\dot{H} > 0$ . Therefore, using the condition (3.4) the equation (4.7) tells us that  $\dot{H} > 0$ . One can also show that for  $0 < q < 1$ , the expression of the scale factor ( $a(\eta) = A(-\eta)^{-q}$ ) gives us  $\dot{H} > 0$ . So, under the condition  $0 < q < 1$  one gets positive acceleration and at the same time one can see that the scale factor grows in cosmic time. This phase of evolution is known as super-inflation [20–22].

## 5 Solutions in the sub-Hubble and super-Hubble scale

Following [24, 25] the general solution of (3.5) can be written as:

$$B_i = \int \varepsilon_{ir}(\vec{k}) \left[ b_r(\vec{k}) B(\eta, k) e^{i\vec{k} \cdot \vec{x}} + b_r^\dagger(\vec{k}) B^*(\eta, k) e^{-i\vec{k} \cdot \vec{x}} \right] \frac{d^3k}{(2\pi)^{3/2}}. \quad (5.1)$$

$b_r$  and  $b_r^\dagger$  are the annihilation and creation operators respectively. Here  $\varepsilon_{\mu r}(\vec{k}) = (0, \vec{\varepsilon}_r(\vec{k}))$  is the polarisation vector and  $r = 1, 2, 3$ .  $\vec{\varepsilon}_1(\vec{k})$ ,  $\vec{\varepsilon}_2(\vec{k})$  are the mutually orthogonal unit vectors also orthogonal to  $\vec{k}$  and  $\vec{\varepsilon}_3(\vec{k})$  is the unit vector along the direction of  $\vec{k}$ . Here we have defined the polarisation vectors slightly differently than in the references [24, 25] (check appendix (A) for discussion). The advantage of doing so is that, instead of redefining the scalar ( $\bar{B} = aB$ ) [24, 25], in Fourier mode one can now directly write the evolution equation (3.5) in terms of the scalar quantity  $B(\eta, k)$  by replacing  $\Delta$  with  $-k^2$ ,

$$B'' + A_1 B' + A_2 k^2 B = 0. \quad (5.2)$$

One can see that the coefficient of  $k$ ,  $\sqrt{A_2}$ , in equation (5.2) is not a constant in time. One can remove the time dependent coefficient of  $k^2$  by redefining the time parameter as  $d\tilde{\eta} = \sqrt{A_2} d\eta$  [26]. Under this change of the time variable, equation (3.5) can be written as

$$B_{,\tilde{\eta}\tilde{\eta}} + \tilde{A}_1 B_{,\tilde{\eta}} + k^2 B = 0, \quad (5.3)$$

where  $\tilde{A}_1 = \frac{1}{2} \frac{A_{2,\tilde{\eta}}}{A_2} + \frac{A_1}{\sqrt{A_2}}$ . Here, in terms of  $\tilde{\eta}$ ,  $A_1$  becomes  $A_1(\tilde{\eta}) = 2\sqrt{A_2} \left( \frac{\varphi_{,\tilde{\eta}}}{\varphi} + \tilde{\mathcal{H}} \right)$ , where  $\tilde{\mathcal{H}} = a_{,\tilde{\eta}}/a$ . One can eliminate  $B_{,\tilde{\eta}}$  term from equation (5.3) by redefining  $B(\tilde{\eta}, k) = \mathcal{B}(\tilde{\eta}, k) f(\tilde{\eta})$ . Substituting this form of  $B$  in equation (5.3) one can eliminate  $\mathcal{B}_{,\tilde{\eta}}$  term by setting its coefficient as zero, which finally gives us

$$f = \exp \left( -\frac{1}{2} \int \tilde{A}_1 d\tilde{\eta} \right), \quad (5.4)$$

$$\mathcal{B}_{,\tilde{\eta}\tilde{\eta}} + \left[ k^2 - \left( \frac{\tilde{A}_{1,\tilde{\eta}}}{2} + \frac{\tilde{A}_1^2}{4} \right) \right] \mathcal{B} = 0. \quad (5.5)$$

From the equation (4.2) one can see that  $\varphi^2 < 8M_{Pl}^2$ , as  $\mathcal{H}$  can not be imaginary. Thus, when  $\frac{8M_{Pl}^2}{\varphi^2}$  is much larger than 2, one can write  $\frac{A_{2,\tilde{\eta}}}{A_2} \approx -2\frac{\varphi_{,\tilde{\eta}}}{\varphi}$ . Therefore, under this condition the expression of  $\tilde{A}_1$  becomes  $\tilde{A}_1 \approx \frac{\varphi_{,\tilde{\eta}}}{\varphi} + 2\tilde{\mathcal{H}}$ . Using the condition (3.4) (which, in terms of  $\tilde{\eta}$ , remains unchanged) the factor  $f$  becomes  $f = \exp \left( -\frac{9}{8} \Delta N \right)$ . Here  $\Delta N \approx \int \tilde{\mathcal{H}} d\tilde{\eta} = \int \mathcal{H} d\eta$  denotes the number of e-folding required for super-inflation. This factor acts as a suppressing factor. The larger the number of e-folding, the larger is the suppressing factor.

Using the expression of  $\tilde{A}_1$  and the condition (3.4) in terms of  $\tilde{\eta}$  equation (5.5) can be expressed in terms of  $\mathcal{H}$  as

$$\mathcal{B}_{,\tilde{\eta}\tilde{\eta}} + \left[ k^2 - \left( \frac{9}{8} \tilde{\mathcal{H}}_{,\tilde{\eta}} + \frac{81}{64} \tilde{\mathcal{H}}^2 \right) \right] \mathcal{B} = 0. \quad (5.6)$$

Using (4.4) and the expression of  $\sqrt{A_2}$  one can write the expression of  $\tilde{\eta}$  in terms of  $\eta$  as:  $-\tilde{\eta} = \frac{(-\eta)^{1-2p}}{1-2p}$ . Therefore, one can see that  $\eta \rightarrow -\infty \implies \tilde{\eta} \rightarrow -\infty$  and  $\eta \rightarrow 0 \implies \tilde{\eta} \rightarrow 0$ , when  $p \ll 1$ . Thus, the expression of the scale factor in terms of  $\tilde{\eta}$  becomes  $a(\tilde{\eta}) = (-\tilde{\eta})^{-m}$ , where  $m = \frac{q}{1-2p}$ . Therefore, in this case also  $0 < m \ll 1$ . Finally, using the expression of  $a(\tilde{\eta})$  the evolution equation (5.6) can be expressed in terms of Bessel differential equation:

$$\mathcal{B}_{,\tilde{\eta}\tilde{\eta}} + \left[ k^2 - \frac{1}{\tilde{\eta}^2} \left( \nu^2 - \frac{1}{4} \right) \right] \mathcal{B} = 0, \quad (5.7)$$



where  $\nu^2 = \frac{1}{4} \left[ 1 + \left( \frac{153}{16} + \frac{9}{2} \tilde{\epsilon} \right) m^2 \right]$ . Where  $\tilde{\epsilon} = 1 - \frac{\tilde{\mathcal{H}}_{,\tilde{\eta}}}{\tilde{\mathcal{H}}}$  is the slow-roll parameter in the standard inflationary scenario. However, it should be noted that in the super-inflationary phase  $\tilde{\epsilon}$  can be  $\sim 1$  or large. In the super-inflationary phase one can achieve acceleration without  $\tilde{\epsilon}$  requiring to be smaller than one.

As  $\nu$  is positive, the solution of the equation can be expressed in terms of the Hankel function of the first and second kind

$$\mathcal{B}_k = \sqrt{-\tilde{\eta}} \left[ a_1(k) H_\nu^{(1)}(x) + a_2(k) H_\nu^{(2)}(x) \right], \quad (5.8)$$

where  $x = -k\tilde{\eta}$ .

In the limit of  $x \gg 1$ , the properties of the Hankel functions are following

$$\begin{aligned} H_\nu^{(1)}(x \gg 1) &\approx \sqrt{\frac{2}{\pi x}} e^{ix - i\frac{\pi}{2}(\nu + \frac{1}{2})}, \\ H_\nu^{(2)}(x \gg 1) &\approx \sqrt{\frac{2}{\pi x}} e^{-ix + i\frac{\pi}{2}(\nu + \frac{1}{2})}. \end{aligned} \quad (5.9)$$

Following the methods used in case of standard inflationary theory, to match with the plane wave solution in the sub-Hubble scale –  $\mathcal{B}_k \sim \frac{e^{ix}}{\sqrt{k}}$  – one can set  $a_2 = 0$  and  $a_1 = \frac{\sqrt{\pi}}{2} e^{i\frac{\pi}{2}(\nu + \frac{1}{2})}$ .

The property of the Hankel function  $H_\nu^{(1)}(x)$ , in the limit of  $x \ll 1$  becomes

$$H_\nu^{(1)}(x \ll 1) \approx \sqrt{\frac{2}{\pi}} e^{-i\frac{\pi}{2}} 2^{(\nu - \frac{3}{2})} \frac{\Gamma(\nu)}{\Gamma(\frac{3}{2})} x^{-\nu}. \quad (5.10)$$

Thus, the solution of  $\mathcal{B}_k$  in the super-Hubble scale can be written as

$$|\mathcal{B}_k| \sim \sqrt{-\tilde{\eta}} x^{-\nu} = \frac{k}{\sqrt{k^3}} (k)^{\frac{1}{2}-\nu} (-\tilde{\eta})^{\frac{1}{2}-\nu}. \quad (5.11)$$

Using the fact that the solutions nearly remains unchanged after Hubble crossing, one can use  $k = \tilde{\mathcal{H}}$  during Hubble crossing and the expression (5.11) can be rewritten as

$$|\mathcal{B}_k| \sim \frac{\tilde{\mathcal{H}}}{\sqrt{k^3}} (k)^{\frac{1}{2}-\nu} (-\tilde{\eta})^{\frac{1}{2}-\nu}. \quad (5.12)$$

Finally, the expression of  $(B)$  can be written as:

$$|B_k| \sim e^{-\frac{9}{8}\Delta N} \frac{\tilde{\mathcal{H}}}{\sqrt{k^3}} (k)^{\frac{1}{2}-\nu} (-\tilde{\eta})^{\frac{1}{2}-\nu}. \quad (5.13)$$

From the expression of  $\nu$  one can identify the spectral index  $n_V$  for vector modes as:

$$n_V = \left( \frac{153}{16} + \frac{9}{2} \tilde{\epsilon} \right) m^2. \quad (5.14)$$

For a small value of  $m$  one can approximate  $\nu \sim \frac{1}{2}$ . Therefore, from equation (5.13) one can see that in the super-Hubble scale the vector modes will be nearly frozen and nearly scale independent similar to the scalar perturbations. It is important to note that there are no observational evidence for scale independent vector perturbation yet. However, super-inflationary phase naturally provides the scale invariance. One can identify that the term

similar to  $\frac{\tilde{\mathcal{H}}}{\sqrt{k^3}}$  also appears in the amplitude of the scalar perturbations, for example one can see reference [27]. But in case of vector modes the amplitude is suppressed by the factor  $(e^{-\frac{9}{8}\Delta N})$  compared to the scalar modes. Here,  $\Delta N$  is the number of e-foldings required for super-inflation. The super-inflationary phase requires new physics which in our case is provided by ELKO. Hence, the super-inflationary phase precedes the standard inflationary phase and the number of e-folding is  $\sim \mathcal{O}(1)$ . The observation of the vector modes in the CMB polarization can restrict the number of e-foldings of super-inflation.

## 6 Conclusion

In this work we have calculated the perturbed energy-momentum tensor of the ELKO. It has been shown that unlike the standard scalar field case, the non-diagonal component (specifically,  $(\eta, i)$ ) of the stress-energy tensor is non-zero. The same component of the Einstein equation gives us the evolution equation of the vector modes ( $B_i$ ) which looks similar to the scalar perturbation equation. We have analytically obtained the vector perturbations for the background evolution satisfied by  $(\varphi' = \frac{1}{4}\varphi\mathcal{H})$ . We have shown that this condition leads to super-inflation ( $\dot{H} > 0$ ).

We have shown explicitly that the vector perturbations are nearly scale invariant and frozen in the super-Hubble scales. However, the amplitude of these perturbations are smaller compared to the scalar perturbation.

In the super-Hubble scale one can look at the behaviour of the solution of (3.3) by setting  $k \rightarrow 0$ . As  $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$ , each  $k_i$  will also be small, i.e.,  $k_i \rightarrow 0$ . As, so far we have not observed any vector modes, one can presume that they have smaller amplitude than the scalar perturbation  $\delta\varphi/\varphi$ . Therefore, in general, one can not ignore the last term in (3.3) in the super-Hubble scale. However, as  $\delta\varphi/\varphi$  is restricted by the observation to be very small, we presume that it is always possible to find a suitable  $k_i$  for which the last term in (3.3) can also be smaller than the first two terms. Under such conditions, in the super-Hubble scale, equation (3.3) can be simplified as

$$B_i'' + A_1 B_i' = 0, \quad (6.1)$$

which tells us that for positive value of  $A_1$ , the solution for  $B_i$  is frozen in the super-Hubble scale for a given initial condition, similar to the scalar perturbation. However, the initial condition is given by the solutions in the sub-Hubble scale, which can bring the  $k$  dependence in the full solutions of (3.3). In this work we have used the condition  $\varphi' = \frac{1}{4}\varphi\mathcal{H}$  which gives us super-inflation. However, one can understand from equation (6.1) that during standard inflation, even when the above condition is violated, the vector modes will be nearly constant in time in the super-Hubble scale. This needs further investigation.

As, usually the vorticity in the perfect fluid follow nearly the similar equation as the vector modes (in this case it will be second order differential equation in time with some additional terms), hence, the vorticity can also be generated in the first order perturbation theory. With the generated vorticity one can also look into the production of large-scale primordial magnetic field. Once the *Planck* polarization results are published, the consequences of this kind of models of vector perturbations can be verified. For the observational possibilities one can look into the Ref. [28].

We know that most of the inflationary models with standard scalar fields can not produce pure vector modes in the first order. Apart from producing scalar perturbations consistent

with observations, the inflationary theory driven by non-standard spinors like ELKO can also produce pure vector modes in the first order. Therefore, the observation of vector modes in the CMB polarization can make this kind of non-standard spinors a potential candidate for inflaton.

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## A Appendix: Polarisation vector in the curved space-time

Following the formalism in the quantum field theory (for example, one can look at [29]) in flat Minkowski space-time, the orthonormality and completeness relations of the polarisation vectors  $\varepsilon_r^\mu(\vec{k})$  can be written as

$$\varepsilon_r(\vec{k}) \varepsilon_s(\vec{k}) = \varepsilon_{r\mu}(\vec{k}) \varepsilon_s^\mu(\vec{k}) = -\zeta_r \delta_{rs}, \quad r = 0, \dots, 3, \quad (\text{A.1})$$

$$\sum_r \zeta_r \varepsilon_r^\mu(\vec{k}) \varepsilon_s^\nu(\vec{k}) = -\eta^{\mu\nu}, \quad (\text{A.2})$$

$$\zeta_0 = -1, \quad \zeta_1 = \zeta_2 = \zeta_3 = 1. \quad (\text{A.3})$$

Here  $\varepsilon_{r\mu}(\vec{k}) = \eta_{\mu\nu} \varepsilon_r^\nu(\vec{k})$ . The above formalism is used in the quantisation of the electro-magnetic field ( $A^\mu$ ) in the Minkowski space-time. One can choose the polarisation vector as  $\varepsilon_r^\mu(\vec{k}) = (0, \vec{\varepsilon}_r(\vec{k}))$ , where  $r = 1, \dots, 3$ .  $\vec{\varepsilon}_1(\vec{k})$ ,  $\vec{\varepsilon}_2(\vec{k})$  are the mutually orthogonal unit vectors also orthogonal to  $\vec{k}$ .  $\vec{\varepsilon}_3(\vec{k})$  is the unit vector along the direction of  $\vec{k}$ . Therefore,  $\vec{\varepsilon}_1(\vec{k})$ ,  $\vec{\varepsilon}_2(\vec{k})$  are also orthogonal to  $\vec{\varepsilon}_3(\vec{k})$ .

To generalise the expressions (A.1), (A.2) and (A.3) in the curved space-time, in reference [24, 25] the authors have chosen to multiply  $1/a$  with all components of the polarisation vector  $\varepsilon_r^\mu(\vec{k})$ . Then one can replace  $\eta^{\mu\nu}$  with  $g^{\mu\nu}$  in (A.2) and the expression (A.3) remains unchanged. *This formalism gives correct solutions of the electro-magnetic vector potential  $A_i$  as one can always absorb the scale factor in the scalar term which appears in the Fourier decomposition of  $A_i$ .* However, this formalism tells us that  $\varepsilon_r^\mu(\vec{k})$  is no longer a function of only  $\vec{k}$ , it also becomes a function of time because of the presence of the scale factor.

Keeping in mind the above points, we propose a slightly different formalism of polarisation vectors to quantize the vector fields of contravariant and covariant form in the curved space-time.

(i) Vector fields of contravariant form  $A^\mu$ : The Fourier decomposition is given as:

$$A^\mu = \int \varepsilon_r^\mu(\vec{k}) \left[ b_r(\vec{k}) A(\eta, k) e^{i\vec{k} \cdot \vec{x}} + b_r^\dagger(\vec{k}) A^*(\eta, k) e^{-i\vec{k} \cdot \vec{x}} \right] \frac{d^3 k}{(2\pi)^{3/2}} \quad (\text{A.4})$$

where,  $\varepsilon_r^\mu(\vec{k}) = (0, \vec{\varepsilon}_r(\vec{k}))$ . The orthonormality and completeness conditions are given as:

$$\varepsilon_r(\vec{k}) \varepsilon_s(\vec{k}) = \varepsilon_{r\mu}(\vec{k}) \varepsilon_s^\mu(\vec{k}) = -\frac{1}{\zeta_r} \delta_{rs}, \quad r = 0, \dots, 3, \quad (\text{A.5})$$

$$\sum_r \zeta_r \varepsilon_r^\mu(\vec{k}) \varepsilon_s^\nu(\vec{k}) = -g^{\mu\nu}, \quad (\text{A.6})$$

$$\zeta_0 = -1/a^2, \quad \zeta_1 = \zeta_2 = \zeta_3 = 1/a^2. \quad (\text{A.7})$$

(ii) Vector fields of covariant form  $A_\mu$ : The Fourier decomposition is given as:

$$A_\mu = \int \varepsilon_{\mu r}(\vec{k}) \left[ b_r(\vec{k}) A(\eta, k) e^{i\vec{k} \cdot \vec{x}} + b_r^\dagger(\vec{k}) A^*(\eta, k) e^{-i\vec{k} \cdot \vec{x}} \right] \frac{d^3 k}{(2\pi)^{3/2}} \quad (\text{A.8})$$

where,  $\varepsilon_{\mu r}(\vec{k}) = (0, \vec{\varepsilon}_r(\vec{k}))$ . The orthonormality and completeness conditions are given as:

$$\varepsilon_r(\vec{k}) \varepsilon_s(\vec{k}) = \varepsilon_{r\mu}(\vec{k}) \varepsilon_s^\mu(\vec{k}) = -\frac{1}{\zeta_r} \delta_{rs}, \quad r = 0, \dots, 3, \quad (\text{A.9})$$

$$\sum_r \zeta_r \varepsilon_{\mu r}(\vec{k}) \varepsilon_{\nu s}(\vec{k}) = -g_{\mu\nu}, \quad (\text{A.10})$$

$$\zeta_0 = -a^2, \quad \zeta_1 = \zeta_2 = \zeta_3 = a^2. \quad (\text{A.11})$$

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